OBLIQUE REFLECTIONS AND UNIMODULAR STRAINS[*]

BY

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Introduction

The object of the present communication is to study the group of unimodular strains about a fixed point in its relation to products of involutory strains.[†] The interest in this group is not merely geometric; rather is it mechanical. The group of strains is fundamental in the theory of elasticity and as such has long been studied by students of both theoretical and applied mathematics. Indeed the different types of strain, such as simple and complex shears, tonics and cyclotonics, have been carefully classified [‡] so that the present investigation has something well known to which to attach itself. The restriction to unimodular or equivoluminal strains is necessitated by the fact that an involutory strain obviously cannot alter (except for sign) the volume of any portion of space, and hence the most general strain that can be obtained as a product of successive involutory strains must leave volume unchanged in magnitude. It may be worth while to note that the ether is a body which possesses this unimodular elastic property, and that the instantaneous strain at a point in an incompressible fluid is similarly characterized.

As the use of multiple algebras in geometric investigations has not been very wide spread, it may be proper at this juncture to enter somewhat upon the question of what algebra should be chosen for the present purpose. Four, at least, suggest themselves immediately. They are: GRASSMANN's point analysis, PEANO's vector analysis, quaternions, and GIBBS's vector analysis.[§] Any of these,

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and others as well, might serve. The first two are more truly projective or strain algebras than the last two. They do not depend so essentially on relations of perpendicularity, which of course are not preserved by strains. Hence they would appear more germane to the subject in hand.* The physicists, however, make great use of relations of perpendicularity in connection with the study of strains — the strain invariants are estimated on perpendicular axes, the strain quadric is referred to such axes, and so on. In short it does appear desirable to take account of perpendicularity.† This is done alike by quaternions and the vector analysis of GIBBS. The former seems nevertheless to be not quite so well suited to the questions in hand as is the latter; for the linear vector function of the quaternion analysis is not treated primarily as an independent multiple quantity with an independent set of algebraic relations but rather as an operator meaningless apart from the vector to which it is applied. Moreover strains have been carefully treated in GIBBS's work ‡ and the treatment is perhaps more easily available for reference than in any other which uses multiple algebra. Hence, everything considered, GIBBS's algebra will be chosen in this instance, although any one of the others would doubtless exhibit at some stage of the work its peculiar advantages over the rest.

It may be taken as known that there are three and only three types of real involutory strains: § 1°, a strain which reverses in direction every vector issuing from the origin; 2°, a strain which reverses in direction vectors parallel to a given line and leaves unchanged vectors parallel to a given plane which does not contain the line; 3°, a strain which reverses in direction vectors parallel to a given plane and leaves unchanged vectors parallel to a given line which does not lie in the plane. The first of these three types of involutory strain will be called *central reflection;* the second, *planar reflection* parallel to a line; the third, *linear reflection* parallel to a plane. The last two types are, so to speak, *dual* in nature — one through a plane parallel to a line, the other through a line parallel to a plane.

These transformations may be expressed as dyadics.‖ Let the dyadics be

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account of the methods of PEANO. See also PEANO, *Formulario Matematico* (editio v) (1906), pp. 165–201. The real difficulty with these algebras is not so much their inappropriateness as the undeveloped state of their linear vector function.

* For this side of the question consult MEHMKE, loc. cit.

† For this side of the question consult PRANDTL, *Ueber die physikalische Richtung in der Vektoranalysis,* Jahresberichte etc, vol. 13 (1904), pp. 436–449. There are some general remarks on both sides of the question by myself in the *Verhandlungen des III. internationalen Mathematiker-Kongresses* (1904), pp. 202–215.

‡ In the pamphlet referred to above and more especially in his *Vector Analysis* edited by myself (1901), chap. 6 (Charles Scribner's Sons, New York). This work will be referred to as *Vector Analysis* in the remaining footnotes or portions of the text.

§ For a full discussion of the types of involutory strains in the projective group and its subgroups see my *Involutory transformations,* etc., previously cited.

‖ *Vector Analysis,* p. 332.

used as prefactors so that

$$\rho' = \Phi \cdot \rho$$

represents the transformation of the position vector ρ into the position vector ρ'. Consider the dyadic Φ to be expressed in terms of any three noncomplanar vectors α, β, γ and their reciprocals α', β', γ' as antecedents and consequents respectively. The identical transformation is represented by the idemfactor which in this case takes the form *

$$I = \alpha\alpha' + \beta\beta' + \gamma\gamma'.$$

Central reflection which merely effects a reversal of all directions is represented by the negative of the idemfactor, that is by

$$-I = -(\alpha\alpha' + \beta\beta' + \gamma\gamma'),$$

and this is the only way in which it may be represented so long as the antecedents are to form a set reciprocal to the consequents.

For oblique planar reflection in the plane of α, β and along the direction of γ, the dyadic

$$\Phi = \alpha\alpha' + \beta\beta' - \gamma\gamma' = I - 2\gamma\gamma'$$

evidently suffices. The analytical form of the dyadic will not be nearly so simple unless the vectors α, β, γ are chosen in the plane and along the direction of the reflection. There, is however, a form which at least in appearance is a little more general than the above and which we will adopt in the future. Assume for trial the form

$$\Phi = I - 2\epsilon\zeta.$$

Any vector ρ perpendicular to ζ becomes

$$\rho' = \Phi \cdot \rho = I \cdot \rho - 2\epsilon\zeta \cdot \rho = \rho.$$

Hence the plane perpendicular to ζ and all components parallel to this plane are left unchanged. Any vector parallel to ϵ becomes

$$\rho' = \Phi \cdot x\epsilon = (1 - 2\zeta \cdot \epsilon)x\epsilon,$$

and this amounts to reversal of direction when and only when $\zeta \cdot \epsilon = 1$.

Hence the general form of a dyadic which shall represent reflection in a plane perpendicular to ζ and along the direction ϵ is

$$\Phi = I - 2\epsilon\zeta, \qquad \zeta \cdot \epsilon = 1.$$

If for any reason some other form of the dyadic should be desired, it is only necessary to express I, ϵ, ζ in terms of any convenient antecedents and consequents.

In like manner a simple inspection shows that the dyadic

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* *Vector Analysis*, p. 288, et seq.

$$\Phi = - \alpha\alpha' - \beta\beta' + \gamma\gamma' = 2\gamma\gamma' - I$$

represents an oblique linear reflection through the line whose direction is γ and parallel to the plane of α, β. This admits of the more general form

$$\Phi = 2\epsilon\zeta - I, \qquad \epsilon \cdot \zeta = 1.$$

It is noteworthy that both the linear and the planar reflections have a fundamental *line* and *plane*, although the part played by the line and plane is related in a sort of dual fashion in the two cases. To sum up the essential points in the analysis we state

THEOREM 1. *The three forms of reflection in the group of strains with their corresponding dyadics are*

$$\begin{aligned}
&\textit{central,} \quad &&\Phi = - I, \\
&\textit{planar,.} \quad &&\Phi = I - 2\epsilon\zeta, \qquad &&\epsilon \cdot \zeta = 1, \\
&\textit{linear,} \quad &&\Phi = 2\epsilon\zeta - I, \qquad &&\epsilon \cdot \zeta = 1,
\end{aligned}$$

where ϵ represents the line that enters into the reflection whether the reflection takes place along it or through it, and where ζ is perpendicular to, that is, represents * *the plane of the reflection whether the reflection takes place through this plane or parallel to it.*

The analytical forms given above have been obtained from the known geometric properties of the reflections which have been proved to be the sole involutory strains. If it were not the intention to make use of the general theory of involutory collineations before entering upon the detail of this paper, the special case of involutory strains which concerns us might have been treated directly as an algebraic problem, and in the following way. Let Φ represent any involutory strain. Then

$$\rho' = \Phi \cdot \rho, \qquad \rho = \Phi \cdot \rho' = \Phi \cdot \Phi \cdot \rho = \Phi^2 \cdot \rho.$$

Hence, as the relation

$$\rho = I \cdot \rho = \Phi^2 \cdot \rho$$

holds for all vectors ρ, it follows that

$$I = \Phi^2.$$

In other words the geometric problem of determining all involutory strains is coextensive with the algebraic problem of finding all dyadics Φ which satisfy the equation $\Phi^2 = I$, that is, which are square roots of the idemfactor. To solve the equation $\Phi^2 = I$, assume the solution in the form

$$\Phi = \Psi + I.$$

Then

$$I = \Phi^2 = (\Psi + I) \cdot (\Psi + I) = \Psi \cdot \Psi + 2\Psi \cdot I + I.$$

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* *Vector Analysis*, p. 46.

Hence

$$\Psi \cdot \Psi + 2\Psi \cdot I = \Psi \cdot (\Psi + 2I) = 0.$$

Now if the product of two dyadics is zero one of them must be zero or one of them must be linear while the other is at most merely planar.* If $\Psi = 0$, then $\Phi = I$, which is involutory although trivially so. If $\Psi + 2I = 0$, then $\Phi = -I$, which is the central reflection. Next suppose that Ψ is linear and write it as $-2\epsilon\zeta$. Then

$$\Phi = I - 2\epsilon\zeta \quad \text{and} \quad \Psi + 2I = 2(I - \epsilon\zeta).$$

The latter must be planar. The condition of planarity is that the third of the dyadic vanish. Hence

$$(I - \epsilon\zeta)_3 = I_3 - \epsilon \cdot I \cdot \zeta = 1 - \epsilon \cdot \zeta = 0,†$$

or $\epsilon \cdot \zeta = 1$ as before. In like manner if $\Psi + 2I$ were linear, assume

$$\Psi + 2I = + 2\epsilon\zeta, \qquad \Phi = 2\epsilon\zeta - I.$$

Then $\Psi = 2(\epsilon\zeta - I)$ must be planar and the application of the condition shows that $\epsilon \cdot \zeta = 1$ again in this case. Hence the problem has been solved by purely algebraic means. The only square roots of the idemfactor are of the form

$$\pm I, \qquad \pm (I - 2\epsilon\zeta) \text{ with } \epsilon \cdot \zeta = 1.$$

The dyadic $\Phi_2 = \frac{1}{2}\Phi {\overset{\times}{\times}} \Phi$ gives the transformation of surfaces.‡ If Φ_2 be calculated it is seen to be

$$\Phi_2 = I \quad \text{if} \quad \Phi = \pm I,$$

or

$$\Phi_2 = 2\zeta\epsilon - I \quad \text{if} \quad \Phi = \pm (I - 2\epsilon\zeta).$$

The change of volume is given by Φ_3. The calculation § of this shows that volumes are reversed in sign by central or planar reflection, but are left unchanged by linear reflection. The results may be stated as

THEOREM 2. *In central reflection surfaces (regarded as vectors) are unchanged while volumes change sign. In planar and linear reflection surfaces (regarded as vectors) suffer linear reflection through a line perpendicular to the plane of the original reflection Φ and along a plane perpendicular to the line of the reflection Φ. Volumes change or keep their sign according as Φ is planar or linear.*

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* *Vector Analysis*, pp. 282–288. This is but a particular case of a theorem in the general theory of matrices and due to SYLVESTER, namely : The nullity of a product of two matrices is not less than the greater of the nullities of the factors nor greater than the sum of the nullities of both factors. See WHITEHEAD, *Universal Algebra*, Vol. 1 (1899), p. 253.

† *Vector Analysis*, p. 331, Exs. 19 and 20. These formulas are important.

‡ *Vector Analysis*, p. 333. The best method of calculating ϕ_2 in this case is by the relation (68) of p. 313; for as ϕ is involutory, $\phi_2 = \pm \phi_c^{-1}$.

§ *Vector Analysis*, p. 333. The value of ϕ_3 may be calculated as above.

1. *The special case in the composition of two reflections.*

One of the first questions to arise is: When are two reflections commutative? It is obvious that whatever be the dyadic Φ,

$$- I \cdot \Phi = \Phi \cdot (- I) = - \Phi$$

and hence that central reflection is commutative not merely with other reflections, but with any strain whatsoever. Consider therefore two reflections neither of which is central. They may be written as

$$\pm (I - 2\epsilon\zeta) \qquad \text{and} \qquad \pm (I - 2\eta\xi),$$

where the sign is plus if the reflection is planar, and minus if it is linear. The condition that the reflections be commutative is

$$\pm (I - 2\epsilon\zeta) \cdot (I - 2\eta\xi) = \pm (I - 2\eta\xi) \cdot (I - 2\epsilon\zeta),$$

which, on expansion, reduces to the equation

$$\zeta \cdot \eta\epsilon\xi = \xi \cdot \epsilon\eta\xi.$$

This equation is satisfied, first if ϵ and η are parallel (as they both may be considered as passing through the origin, they may be taken as actually collinear) and ζ and ξ are parallel at the same time; second, if $\zeta \cdot \eta = 0$ and $\zeta \cdot \epsilon = 0$. The latter case is obvious, and the former becomes so when it is remembered that a single dyad possesses but five degrees of freedom, of which four are here satisfied by the relations of collinearity and the fifth by the equality in magnitude which results from the homogeneity of the equation. The geometric relation of the reflections in the first case needs no discussion; in the second case the equation $\zeta \cdot \eta = 0$ shows that η is perpendicular to ζ. Now ζ represents the plane of the reflection. Hence the line η lies in the plane perpendicular to ζ, that is, in the plane of the reflection. To sum up:

THEOREM 3.   *That two reflections be commutative it is necessary and sufficient that: 1°, one of them be central; 2°, their lines and planes be respectively coincident; or 3°, the lines of each lie in the plane of the other.*

To find the result of the product of two commutative reflections note first that the product of a central reflection into a planar or linear reflection gives the other one of the two. For the product of two reflections, neither of which is central, we have

$$\pm (I - 2\epsilon\zeta) \cdot (I - 2\eta\xi) = \pm (I - 2\epsilon\zeta^{\cdot} - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi).$$

In case ϵ and ζ are respectively parallel to η and ξ they may be taken equal to them, for the relations $\epsilon \cdot \zeta = \eta \cdot \xi = 1$ show that the angle between the pairs of vectors ϵ, ζ and η, ζ are the same and hence that the vectors are either equal or opposite, which amounts to the same thing in this problem. Hence the product

is $\pm I$. In case $\zeta \cdot \eta = \xi \cdot \epsilon = 0$ the product is

$$\pm (I - 2\epsilon\zeta - 2\eta\xi)$$

with the relations

$$\epsilon \cdot \zeta = 1, \qquad \eta \cdot \xi = 1, \qquad \epsilon \cdot \xi = 0, \qquad \zeta \cdot \eta = 0.$$

Hence

$$\zeta \times \xi \cdot \epsilon \times \eta = \zeta \cdot \epsilon \xi \cdot \eta - \xi \cdot \epsilon \xi \cdot \eta = 1.$$

The vectors ϵ, η, $\zeta \times \xi$ and ζ, ξ, $\epsilon \times \eta$ therefore satisfy the relations for reciprocal systems. If they be chosen as α, β, γ and α', β', γ', the product takes the form

$$= (I - 2\alpha\alpha' - 2\beta\beta') = = (I - 2\gamma\gamma').$$

The geometric meaning of the results may be summed up in a theorem in which it will be convenient to put some additional but obvious statements concerning the converse problem.

THEOREM 4. *Two commutative reflections compound into the identical transformation when and only when they are identical with one another. And conversely the identical transformation may be resolved in ∞^4 ways into the product of reflections which are identical. Two commutative reflections compound into a central reflection when and only when one is planar and the other is linear with the same fundamental line and plane. Conversely the resolution may be effected in ∞^4 ways into such reflections. Two commutative reflections compound into a planar reflection when and only when they are either central and linear or planar and linear; in the former case the line and plane remain the same, in the latter case the plane of the resultant reflection is the plane of the two lines of the component reflections and the line of the resultant reflection is the line of intersection of the planes of the components. Conversely the resolution of a planar reflection into two commutative reflections may be accomplished in ∞^2 ways by choosing the lines of the desired component reflections at random in the plane of the given reflection and taking the planes of the components through these lines and the line of the given reflection. Two commutative reflections compound into a linear reflection when and only when they are either central and planar or both planar or both linear; in the first case the line and plane remain the same, in the latter cases the line is the intersection of the planes of the components and the plane is the plane through the two lines of the components. A converse similar to the above holds good.*

The analysis has shown that the product of two commutative reflections is a reflection and has furnished specific information as to what reflection the product is. If it had been a matter of no importance to determine the resultant reflection, the fact that it was a reflection could have been inferred as a special case of the general theorem: If the product of two involutory transformations

is commutative, the product is itself involutory.   This general theorem follows from the symbolic equations

$$t_1 t_2 = t_2 t_1 = t_2^{-1} t_1^{-1} = (t_1 t_2)^{-1},$$

where t_1 and t_2 are involutory.   A similar set of equations,

$$t_1 t_2 = (t_1 t_2)^{-1} = t_2^{-1} t_1^{-1} = t_2 t_1,$$

shows that conversely: If the product of two involutory transformations is involutory, the product is commutative.   This enables us to omit the word commutative from the above theorem.   Hence

THEOREM 5.   *The word commutative may be deleted from theorem* 4 ; *for the compositions and resolutions there indicated exist when and only when the component reflections are commutative, that is, when and only when they satisfy the conditions of theorem* 3.

By *definition* two reflections, planar or linear, will be said to be *complanar* when their fundamental planes coincide, and *collinear* when their fundamental lines coincide, irrespective of whether the reflections belong to the same type or not.   It has been seen that two reflections which are both complanar and collinear compound into the identical transformation or into the central reflection according as they are of the same or of different types.   The next question is concerning the composition of reflection which are either complanar or collinear, but not both.

The product of two complanar planar reflections may be written as

$$(I - 2\epsilon\zeta) \cdot (I - 2\eta\zeta), \qquad \epsilon \cdot \zeta = \eta \cdot \zeta = 1.$$

This is immediately reducible to

$$I + 2(\epsilon - \eta)\zeta.$$

The effect of applying this dyadic to any vector ρ is

$$\rho' = \rho + 2(\epsilon - \eta)\zeta \cdot \rho.$$

The condition $\epsilon \cdot \zeta = \eta \cdot \zeta$ gives $(\epsilon - \eta) \cdot \zeta = 0$.   Hence $\epsilon - \eta$ is perpendicular to ζ, or, in other words, lies in the plane represented by ζ.   Hence the transformation consists in adding to any vector a component in the definite direction $\epsilon - \eta$ and of magnitude proportional to the component of that vector perpendicular to the plane represented by ζ.   Points in the plane represented by ζ are unchanged in position.

Now by definition a special simple shear * is a transformation which leaves a plane fixed point for point and which moves points not in this plane along lines

* *Vector Analysis*, p. 363.   Note that in this case and those which follow, the coefficient of I is 1.   This is necessitated by the fact that for any transformation compounded of reflections there is no dilatation.

parallel to a given direction in the plane by an amount proportional to their distance above the plane. Thus a special simple shear is completely specified when its fixed plane, a direction in the plane, and a number which gives the amount of motion in that direction per unit distance from the plane is given. This amount is easy to determine in the above case. Analytically it is $2(\epsilon - \eta)$: for ζ may be taken as a unit vector perpendicular to the common plane, and then, since ρ must be taken of unit length in the direction of ζ if the amount of shift at unit distance from the plane be desired, we have $\xi \cdot \rho = 1$. Now the relations $\epsilon \cdot \zeta = \eta \cdot \zeta = 1$ show that ϵ and η terminate in a plane at unit distance above the fixed plane. Hence

THEOREM 6. *The product of two complanar planar reflections is a special simple shear parallel to the common plane and in the direction of the intersection of this plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first* reflection cuts a plane at a unit distance from the fixed plane to the point where the line of the second* reflection cuts this plane.*†

Two remarks will be sufficient to lead to two more theorems. In the first place the identity

$$(I - 2\epsilon\zeta) \cdot (I - 2\eta\zeta) = (2\epsilon\zeta - I) \cdot (2\eta\zeta - I)$$

shows that no new work need be done on complanar linear reflections. In the second place, it should be noted that any simple shear of the type above considered may be written in the form

$$I + \gamma\beta, \qquad \gamma \cdot \beta = 0.$$

Now if two reflections be chosen with a common plane perpendicular to β and if the choice of the lines ϵ and η be made to conform to the relations

$$\gamma = 2(\epsilon - \eta), \qquad \epsilon \cdot \beta = \eta \cdot \beta = 1,$$

it is evident that the shear has been resolved into two complanar planar reflections. Moreover the choice of the direction of either ϵ or η is arbitrary — the choice of either one confines the other to a perfectly definite direction in the plane of the one first chosen and of γ. Hence

THEOREM 7. *The product of two complanar linear reflections is a special simple shear parallel to the common plane in the direction of the intersection of this*

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* It should be noted that the *first* and *second* reflections are respectively $I - 2\eta\zeta$ and $I - 2\epsilon\zeta$, that is, the order of the reflections is necessarily the opposite of that in which the product of the dyadics is written. The reason for this is obvious.

† The value of this amount may be expressed trigonometrically by direct translation from the vector formulas. If $I - 2\epsilon\zeta$ be written as $I - 2\epsilon_1\zeta_1 \sec(\epsilon\zeta)$, where ϵ_1 and ζ_1 are unit vectors, the evaluation of the amount of shear gives

$$\sqrt{\sec^2(\epsilon\zeta) + \sec^2(\eta\zeta) - 2\sec(\epsilon\zeta)\sec(\eta\zeta)\cos(\epsilon\eta)}.$$

plane with the plane of the two lines of the reflections and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where the line of the first reflection cuts a plane at unit distance from the fixed plane to the point where the line of the second reflection cuts this plane.

THEOREM 8.   *A special simple shear may be resolved in ∞^2 ways into the product of two planar or two linear reflections which are complanar with the fixed plane of the shear, and the line of one reflection may be chosen at random.*   The line of the other reflection will then be uniquely determined by the conditions stated in theorems 6 and 7.*

The discussion may be carried on in similar manner for the case of collinear reflections.   The product

$$(I - 2\varepsilon\zeta)\cdot(I - 2\epsilon\xi) \qquad \epsilon\cdot\zeta = \epsilon\cdot\xi = 1,$$

reduces to the form

$$I + 2\epsilon(\xi - \zeta).$$

The vector ϵ is unchanged because of the relations

$$I\cdot\epsilon = \epsilon, \qquad \xi\cdot\epsilon - \zeta\cdot\epsilon = 0.$$

The vector $\xi \times \zeta$ is likewise unchanged because of the relations

$$I\cdot\xi \times \zeta = \xi \times \zeta, \qquad \xi\cdot\xi \times \zeta = \zeta\cdot\xi \times \zeta = 0$$

Hence points in the plane of ϵ and $\xi \times \zeta$ remain fixed.   This is the plane of the shear which the product obviously represents.   The result of applying the shear to any vector ρ is

$$\rho' = I\cdot\rho + 2\epsilon(\xi - \zeta)\cdot\rho = \rho + 2\epsilon(\xi - \zeta)\cdot\rho.$$

Hence the direction of the shear is ϵ.   The vector which is perpendicular to the plane of the shear is $\epsilon \times (\xi \times \zeta)$ or $\xi - \zeta$.   The relations $\epsilon\cdot\zeta = \epsilon\cdot\xi = 1$ enable one to read off the amount of the shear as twice the distance between the points where the vectors ξ and ζ meet a plane at unit distance from the origin and perpendicular to ϵ.   It is, however, more convenient to note that

$$\sigma = \frac{\xi - \zeta}{|\xi - \zeta|} - \frac{(\xi - \zeta)\cdot\xi}{|\xi - \zeta|}\,\epsilon + y\xi \times \zeta$$

and

$$\tau = \frac{\xi - \zeta}{|\xi - \zeta|} - \frac{(\xi - \zeta)\cdot\zeta}{|\xi - \zeta|}\,\epsilon + y\xi \times \zeta$$

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* The only exception to the at-random-ness of the choice of this line is that it shall not lie in the plane of the shear.   It has been stated previously, however, that this plane is the plane of the reflections, and hence the line could not lie in this plane.   Frequently this same state of affairs occurs.   We shall therefore make the convention that at random, applied to the choice of a line or plane of a reflection, means at random except for such positions as make the line and plane of the reflection occupy this position of coincidence, which would render the reflection meaningless.

are two vectors terminating in a plane at unit distance from the fixed plane of the shear, that they lie respectively in the planes represented by ξ and ζ, and that their difference is precisely the magnitude and direction of the shear. Hence, adding a few obvious propositions, we may state

THEOREM 9. *The product of two collinear reflections, be they both planar or both linear, is a special simple shear parallel to the plane that contains their common line and the line of intersection of their planes and in the direction of their common line and of amount equal (per unit distance from the fixed plane) to twice the distance from the point where a line parallel to and at a unit distance above the fixed plane cuts the plane of the first reflection, to the point where it cuts the plane of the second reflection. And conversely any such shear may be resolved in ∞^2 ways into the product of two collinear reflections ; for the common line may be chosen as the direction of the shear, and the line of intersection of the planes may be taken as any line in the plane of the shear, and the two planes of the reflections may be so set as to cut off one half the amount of the shear upon a line parallel to the line of the shear at unit distance from the plane of the shear.*

If two complanar or collinear reflections one of which is planar and the other linear be compounded the result is of the form

$$ -(I - 2\epsilon\zeta)\cdot(I - 2\eta\zeta) \qquad \text{or} \qquad -(I - 2\epsilon\zeta)\cdot(I - 2\epsilon\xi). $$

Apart from the initial negative sign this is merely a shear, as before. The negative sign reverses all the directions in space. A transformation which is in all respects like a shear except that all directions in space have been reversed may be called a *perverted* [*] shear. Then

THEOREM 10. *The relation of two complanar or collinear reflections, of which one is planar and the other linear, to the perverted shear is the same as the relation of two like complanar or collinear reflections to the (unperverted) shear. See theorems 6–9.*

There have been found a number of ways in which a special simple shear may be resolved into the product of two reflections. The question arises whether all the possibilities have been enumerated. The corresponding question in the case of resolving a reflection into two reflections was answered in theorem 5 by means of general equations in the theory of involutory transformations. This method is not available here. Collineations may, however, be classified, in the simpler cases, as essentially (projectively) different according as they have similar or dissimilar characteristic equations. This method has been carried out in detail for strains in the *Vector Analysis*, pages 356–367. The facts are in part these : Any dyadic Φ satisfies identically a definite cubic equation [†]

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[*] Analogously to the nomenclature of GIBBS. *Vector Analysis*, p. 337.

[†] In general if ϕ denote a dyadic representing a strain in n dimensions and if ϕ_s, ϕ_{2s}, $\cdots$, ϕ_n be its invariants, the dyadic satisfies identically the equation of the nth order

$$ \phi^n - \phi_s\phi^{n-1} + \phi_{2s}\phi^{n-2} \cdots + (-1)^n\phi_n I = (\phi - a_1 I)\cdot(\phi - a_2 I)\cdot\cdots\cdot(\phi - a_n I) = 0 $$

$$\Phi^3 - \Phi_S \Phi^2 + \Phi_{2S} \Phi - \Phi_3 = 0$$

or

$$(\Phi - aI) \cdot (\Phi - bI) \cdot (\Phi - cI) = 0,$$

where the quantities a, b, c are roots of the cubic equation

$$x^3 - \Phi_S x^2 + \Phi_{2S} x - \Phi_3 = 0.$$

If two or three roots of this equation are equal the dyadic may satisfy an equation of lower degree. The equation of lowest degree which the dyadic satisfies is called the *characteristic* equation. There are six distinct types of characteristic equation for strains * (if no distinction is made between real and imaginary roots of the scalar cubic). To test a given strain it is merely necessary to find what type of characteristic equation it has.

In the case of the special simple shear the characteristic equation is of the second order and takes the form

$$(\Phi - I)^2 = 0.$$

The roots of the cubic are all equal to unity (for we are considering merely the case where there is no stretching modulus), and the cubic is

$$x^3 - 3x^2 + 3x - 1 = 0.$$

For the perverted shear the roots are all equal to -1 instead of $+1$ and the characteristic equation becomes

$$(\Phi + I)^2 = 0,$$

while the scalar cubic is

$$x^3 + 3x^2 + 3x + 1 = 0.$$

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where a_1, a_2, $\cdots$, a_n are the roots of a scalar equation of the nth degree. To classify the strain it is convenient to classify first all different sorts of sets of roots which the scalar n-ic can have, according to their multiplicity, and then to subclassify according to the form of the characteristic equation. Thus in the case $n = 4$ the different types are : a, b, c, d ; a, b, c, c ; a, a, c, c ; a, a, a, b ; a, a, a, a. In the first case the characteristic equation is determined as a product of four factors : $\phi - aI$, $\phi - bI$, $\phi - cI$, $\phi - dI$. Not so, however, in the other cases. In the second case, the factor $\phi - cI$ may occur twice, leaving the equation of the nth degree, or it may occur only once, reducing the degree the $(n-1)$st. Similarly in the third case there may be four factors to the characteristic equation, or one of the factors may drop out, or two of them may drop out, reducing the equation to $(\phi - aI) \cdot (\phi - cI)$. There are thus three types of strain corresponding to this set of roots for the scalar n-ic. If all the types be counted up, it appears that there are 13 types of characteristic equation at most. That each of these types exists, is easily shown by actually writing down the expression for it. This classification is not complete except when $n = 2$ or 3 ; for larger values of n additional criteria must be adduced. Thus when $n = 4$, there are 14 types of strain, but only 13 types of characteristic equations, and when $n = 5$ these numbers are 27 and 24 respectively. This difficulty need not concern us here.

* *Vector Analysis*, p. 366. The first two types there given differ only by the fact that in one case two of the roots are conjugate imaginaries. Geometrically this is an essential difference analytically it is unimportant.

This case we will not take up in detail. The entire treatment resembles too closely that for the (unperverted) shear.

Consider any reflection

$$\Phi = \pm (I - 2\epsilon\zeta), \quad \Phi_2 = 2\zeta\epsilon - I, \quad \Phi_S = +1, \quad \Phi_{2S} = -1, \quad \Phi_3 = +1.$$

The invariants Φ_S, Φ_{2S}, Φ_3 which are the coefficients of the scalar cubic may be computed for a product of dyadics by means of the formulas *

$$(\Psi \cdot \Omega)_2 = \Psi_2 \cdot \Omega_2, \quad (\Phi \cdot \Omega)_3 = \Phi_3 \Omega_3.$$

Hence the scalar cubic for the product

$$\Phi = \pm (I - 2\epsilon\zeta) \cdot (I - 2\eta\xi) = \pm (I - 2\epsilon\zeta - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi)\dagger$$

has the form

$$x^3 \mp (4\zeta \cdot \eta\epsilon \cdot \xi - 1)x^2 + (4\epsilon \cdot \xi\zeta \cdot \eta - 1)x \mp 1 = 0.$$

If this product is to be a shear of the type we have been considering, it is necessary that

$$\Phi_S = 3 = \pm (4\zeta \cdot \eta\epsilon \cdot \xi - 1), \quad \Phi_{2S} = 3 = 4\epsilon \cdot \xi\zeta \cdot \eta - 1, \quad \Phi_3 = 1 = \pm 1.$$

The first or last of these relations shows that the two reflections must be of the same type. That is, the upper sign must hold throughout. Since the upper sign holds, form the difference

$$\Phi - I = 2(\epsilon\zeta + \eta\xi - 2\epsilon\xi\zeta \cdot \eta).$$

To multiply out the expression $(\Psi - I)^2$ and hence obtain the condition for a simple shear would be long — a shorter method is desirable. Now if the product $\Psi \cdot \Omega$ of two dyadics is zero, either Ψ or Ω must be linear. Hence in this case $\Phi - I$ is linear. For this the necessary and sufficient condition is that $(\Phi - I)_2$ vanish.‡   But

$$(\Phi - I)_2 = 4\epsilon \times \eta\zeta \times \xi = 0.$$

Hence either ϵ and η are collinear or ζ and ξ are collinear; that is, the reflections are either collinear or complanar.   It has already been seen that they cannot be complanar and collinear.   Hence

THEOREM 11.   *The necessary and sufficient condition that a special simple shear be resoluble into two reflections is that the reflections be of the same type, either linear or planar, and that they be complanar or collinear, but not both. The resolution may then be accomplished in ∞^2 ways as specified in theorems 6–9; similar results hold for the perverted shear.*

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* *Vector Analysis*, p. 312.

† Here and throughout the following work the upper signs belong together and the lower signs belong likewise together, forming only two possible sets.

‡ *Vector Analysis*, p. 315.

Another case where the roots of the scalar cubic are all equal to unity is that of the complex shear of which the characteristic equation is

$$(\Phi - I)^3 = 0.$$

This shear may be reduced to the canonical form *

$$I + \alpha\beta' + \beta\gamma'.$$

The geometric properties of the transformation are these: There is a direction, the direction α, which is unchanged. Vectors parallel to α do not change either in magnitude or in direction. Through the line there is a plane, the plane of α and β, in which points move parallel to α by an amount proportional to their distance from the fixed line α. Through the line there is a second plane, the plane of α and γ, the points of which move parallel to β by an amount proportional to their distance from the line α. It should be noted, however, that the points of any plane passing through the line α, say the plane of α and δ, move parallel to a certain direction in the plane of α and β, namely the direction

$$\alpha\beta' \cdot \delta + \beta\gamma' \cdot \delta.$$

Thus the reduction of the complex shear to the canonical form may be accomplished in a single infinity of ways. A complex strain may therefore be specified by giving its fixed line a fixed plane through this line and a number which expresses the shift parallel to the line (per unit distance from the line), and a second plane through the fixed line with the direction and amount of shift (per unit distance from the line) of the points is this plane.

In distinction from the complex shear, there is the perverted complex shear, which is the combination of the shear with reversal of all directions in space. The characteristic equation of the perverted complex shear is

$$(\Phi + I)^3 = 0$$

and its scalar cubic is
$$x^3 + 3x^2 + 3x + 1 = 0.$$

This case is so similar to that of the (unperverted) shear that it need not be taken up in detail.

Consider then the product of two reflections. Just as before, it appears that the reflections must be of the same type, either planar or linear, if their product is to be a complex shear. The upper sign therefore holds. The value of the invariant Φ_{2S} is

$$\Phi_{2S} = 3 = 4\epsilon \cdot \xi\zeta \cdot \eta - 1.$$

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* *Vector Analysis*, pp. 365-7.

Hence

$$\zeta \cdot \eta \epsilon \cdot \xi = 1 \,.$$

But the relations

$$\epsilon \cdot \zeta = \eta \cdot \xi = 1 \quad \text{and} \quad \epsilon \times \eta \cdot \zeta \times \xi = \epsilon \cdot \zeta \eta \cdot \xi - \eta \cdot \xi \epsilon \cdot \xi$$

show that the condition reduces to

$$\epsilon \times \eta \cdot \zeta \times \xi = 0 \,.$$

And in this equation neither $\epsilon \times \eta$ nor $\zeta \times \xi$ can vanish, or the product would reduce to the case of a simple shear previously considered.   Hence

THEOREM 12.   *The necessary and sufficient condition that the product of two reflections be a complex shear is that the reflections be of the same type and that the lines of the two reflections be complanar with the intersection of the planes of the reflections without the reflections themselves being either complanar or collinear. An analogous result holds for perverted complex shears, the only difference being that the reflections must be of different type.*

Further to discuss the transformation write the product in the form

$$\Phi = I - 2\,(\eta - \zeta \cdot \eta \epsilon)\,\xi - 2\epsilon\,(\zeta - \zeta \cdot \eta \xi) \,.$$

Here

$$(\eta - \zeta \cdot \eta \epsilon)\cdot \xi = \eta \cdot \xi - \zeta \cdot \eta \epsilon \cdot \xi = 0 \,,$$

$$\epsilon \cdot (\zeta - \zeta \cdot \eta \xi) = \epsilon \cdot \zeta - \zeta \cdot \eta \epsilon \cdot \xi = 0 \,,$$

$$(\eta - \zeta \cdot \eta \epsilon)\cdot(\zeta - \zeta \cdot \eta \xi) = 0 \,.$$

Hence the vectors $\eta - \zeta \cdot \eta \epsilon$, ϵ, ξ, $\zeta - \zeta \cdot \eta \xi$ satisfy the relations which are characteristic of the vectors $\alpha, \beta, \beta', \gamma'$ that enter into the canonical expression for the complex shear (with the exception of the relation $\beta \cdot \beta' = 1$ that depends on the magnitudes of the vectors).   The vector $\eta - \zeta \cdot \eta \epsilon$ which corresponds to α is the fixed direction of the shear.   By inspection, however, the vector $\zeta \times \xi$ is unchanged — as is geometrically obvious.   And in fact it is evident, on expanding, that

$$(\eta - \zeta \cdot \eta \epsilon) \times (\zeta \times \xi) = 0 \,.$$

The direction ϵ, which is the line of the second reflection, is the other fundamental line in the fixed plane.   In particular a vector $a\zeta \times \xi + b\epsilon$ becomes

$$a\zeta \times \xi + b\epsilon + 2b\left(\epsilon - \frac{1}{\zeta \cdot \eta}\eta\right) \,.$$

Hence the amount of motion of the points in the fixed plane is

$$2b\left(\epsilon - \frac{1}{\zeta \cdot \eta}\eta\right).$$

Moreover, since $\epsilon \cdot \zeta = 1$, ϵ may be regarded as terminating in the plane determined by ζ, and if η be regarded as terminating in this same plane, $\eta \cdot \zeta = 1$. Then the additional term which represents the motion in the fixed plane reduces to $2b(\epsilon - \eta)$, with the interpretation that the points move parallel to the fixed line $\zeta \times \xi$ by an amount equal (per unit distance from that line) to twice the distance from the point where the first line η cuts a line at a unit distance from that line to the point where the second line ϵ cuts the same line.

The expression for the product shows that any vector ρ in the plane of $\zeta \times \xi$ and ξ suffers no change parallel to the fixed line $\zeta \times \xi$ but is altered by the amount $-2\epsilon\zeta \cdot \rho$ in the direction ϵ. By virtue of the relation $\epsilon \cdot \zeta = 1$ the equation

$$-\epsilon\zeta \cdot \rho = (\epsilon \times \rho) \times \zeta - \rho$$

is identical and shows that the change $-2\epsilon\zeta \cdot \rho$ is twice the distance from the extremity of ρ to the point where a line parallel to ϵ and passing through this extremity cuts the plane represented by ζ. To sum up we have

THEOREM 13.   *The product of two planar or two linear reflections which are neither complanar nor collinear but so related that their two lines and the line of intersection of their two planes are complanar gives rise to a complex shear of which the fixed line is the line of intersection of the two planes, and of which the fixed plane is the common plane of the lines ϵ, η, $\zeta \times \xi$. The amount of the shift parallel to the fixed line and in the fixed plane is equal (per unit distance from the fixed line) to twice the distance from the intersection of the first line η to the intersection of the second line ϵ with a line lying in the fixed plane at a unit distance from the fixed line. The plane in which there is no shift parallel to the fixed line is the plane of the first reflection, and points in this plane are shifted parallel to ϵ by an amount* equal (per unit distance from the fixed line) to twice the distance from a point in the plane of the first reflection and at a unit distance from the fixed line to the point in the plane of the second reflection where a line through this point and parallel to ϵ cuts the plane of the second reflection. An analogous result for the perverted complex shear.*

THEOREM 14.   *Conversely any complex shear may be resolved into the product of two planar or two linear reflections by choosing the lines of the two reflections in the fixed plane of the shear and subject to the sole restriction that the distance of the*

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* The deduction of the trigonometric relations that express the magnitude of the shear is left to the reader.

points where the first and second line respectively cut the line at a unit distance from the fixed line of the shear shall be equal to one-half the amount of shift (per unit distance from the fixed line) in the fixed plane; and by choosing the two planes of the reflections to pass through the fixed line of the shear in such a way that the first plane coincides with the plane whose points are shifted parallel to the line of the second reflection and that the second plane be distant from the first by an amount measured along a line parallel to the second line and through a point at a unit distance from the fixed line equal to one-half the amount of the shift due to the shear. This resolution may therefore be accomplished in ∞^1 ways and not in ∞^2 ways.

The case in which the lines of the reflections lay in the planes, each of the other, gave rise to commutative reflections which compound into another reflection. If the scalar cubic and the characteristic equation had been called into play, it would have appeared that this case corresponds to the roots $-1, -1, +1$ or $-1, +1, +1$ and to the characteristic equations

$$(\Phi - I) \cdot (\Phi + I) = 0 \qquad \text{or} \qquad (\Phi + I) \cdot (\Phi + I) = 0,$$

according as the resulting reflection was linear or planar. There arises then the question: What if only one of the lines lie in the plane of the other reflection? In this case either $\zeta \cdot \eta = 0$ or $\xi \cdot \epsilon = 0$; but not both. There is no need of giving the details of the computation. The result shows that the characteristic equation is of the third order and takes the form

$$(\Phi - I) \cdot (\Phi + I)^2 = 0 \qquad \text{or} \qquad (\Phi + I) \cdot (\Phi - I)^2 = 0,$$

according as the roots are the first or second of the above set.

This sort of equation betokens a simple shear * (non-special). The general shear of this type is

$$a\alpha' - p(\beta\beta' + \gamma\gamma') - \gamma\beta' \qquad \text{or} \qquad a\alpha' + p(\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

Here, in the first case, there is a fixed direction α. Through this direction pass two planes, of which one, the plane of α and γ, is fixed as a whole, though if vectors are resolved parallel to α and γ the γ-components are stretched in the ratio $-p : 1$ and the α-components in the ratio $a : 1$. The points of the other plane, that of α and β, have a compound motion. The vectors which denote the points are stretched along α and β and take on an increment parallel to γ of amount proportional to their components along β. The plane of β and γ is stretched in the ratio $p : 1$ and the points move out of it by an amount proportional to their β components, the motion being in the direction γ. The vector β may be chosen at will in the plane of β and γ. In the case of the above roots the shear takes the form

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* *Vector Analysis*, pp. 366-7.

$$\alpha\alpha' - (\beta\beta' + \gamma\gamma') - \gamma\beta' \quad \text{or} \quad -\alpha\alpha' + (\beta\beta' + \gamma\gamma') + \gamma\beta'.$$

The latter is merely a perversion of the first.

If the line of the first reflection lies in the plane of the second, $\zeta \cdot \eta = 0$, and the product takes the form (we shall consider, as usual, the first or unperverted case)

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi.$$

It is obvious that the line $\zeta \times \xi$ of intersection of the two planes plays the rôle of the fixed line α; the line η, namely that which lies in the plane of other reflection, plays the rôle of γ, parallel to which the shear takes place; the plane of the two lines ϵ and η, that of the plane of β and γ. The amount of the shear suffered by ϵ is

$$-2\eta\xi \cdot \epsilon, \quad \text{but} \quad (\epsilon - \xi \cdot \epsilon\eta) \cdot \xi = 0;$$

which shows that this amount is twice the distance from the plane of ϵ and α to the plane ξ, if that distance be measured parallel to η.

If it is the line of the second reflection that lies in the plane of the first, the product becomes

$$\Phi = I - 2\epsilon\zeta - 2\eta\xi + 4\zeta \cdot \eta\epsilon\xi, \quad \epsilon \cdot \xi = 0.$$

The line of intersection $\zeta \times \xi$ of the two planes still corresponds to the fixed line α. The line ϵ is now the line parallel to which the shear takes place. Thus· in either case it is the line which lies in the plane of the other reflection that determines the direction of the shear. The plane corresponding to the plane of β and γ is still the plane of the two lines. The amount of shear suffered by η is

$$2\epsilon\zeta \cdot \eta, \quad \text{but} \quad (\eta - \epsilon\zeta \cdot \eta)\zeta = 0;$$

which shows that this amount[*] is twice the distance from the plane ζ to the plane of η and α, if the distance be measured parallel to ϵ.

Conversely any simple shear all of whose roots are numerically equal to unity (but not all equal algebraically) may be resolved in ∞^1 ways into the product of two reflections. This may be accomplished by choosing the planes of the reflections through the fixed line and one of them coincident with the fixed plane. The line of the other reflection must then be chosen in this plane and along the direction of the shear parallel to the plane. The line of the other reflection may then be taken at will in a definite plane, after which the final plane is wholly determined by the magnitude of the shear — but differently according as

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[*] The trigonometric values are again left to the reader.

it is the line of the first reflection which is to lie in the plane of the second or vice versa. It will not be necessary to specify all these relations in a theorem or to prove, as may readily be done, that these are the only possible resolutions. It will be sufficient to state

THEOREM 15. *The necessary and sufficient condition that two reflections compound into a simple (non-special) shear of which the roots are numerically equal to unity is that the line of one of the reflections lie in the plane of the other. And conversely any such shear may be resolved into two reflections in ∞^1 ways.*

2. *The general case of the composition of two or three reflections.*

The general product of two reflections and the scalar cubic associated therewith have been given on page 282. The form of the cubic leads to

THEOREM 16. *If a dyadic Φ is the product of two roots of the idemfactor the conditions*

$$\Phi_{2S} = \pm \Phi_S, \qquad \Phi_3 = \pm 1$$

hold. The scalar cubic is a reciprocal equation with one of the three roots equal to ± 1 and with the product of the other two roots equal to $+1$.

As one of the roots of the cubic is ± 1, the equation may be written

$$(x \mp 1)\left[x^2 \mp (\Phi_{2S} - 1)x + 1\right] = 0.$$

The condition that the roots of the second factor be equal is

$$\Phi_{2S} - 1 = \pm 2, \qquad \Phi_{2S} = 3 \text{ or } -1.$$

The three roots are then some combination selected from ± 1, ± 1, ± 1. Geometrically these cases correspond to the identical transformation, the reflections, and the two shears — all of which have been treated in detail in § 1. If the coefficient of the first degree term in the quadratic factor be numerically decreased from the value which gives the equal roots, the roots of the resulting quadratic will be imaginary, whereas if that coefficient be increased the roots will be real. Hence the inequalities

$$|\Phi_{2S} - 1| > 2, \quad \Phi_{2S} > 3 \ \text{ or } \ \Phi_{2S} < -1, \quad \epsilon \cdot \xi\zeta \cdot \eta < 0 \ \text{ or } \ \epsilon \cdot \xi\zeta \cdot \eta > 1$$

obtain when two roots of the cubic are real, and the dyadic reduces to a tonic; while the inequalities

$$|\Phi_{2S} - 1| < 2, \qquad -1 < \Phi_{2S} < 3, \qquad 0 < \epsilon \cdot \xi\zeta \cdot \eta < 1$$

hold when two roots are imaginary and the dyadic reduces to the cyclotonic.

Just as in the case of shears, it will be necessary to distinguish between perverted and unperverted tonics and cyclotonics. The same convention will be adopted, namely, that if the value of Φ_3 is $+1$ so that one of the roots of the cubic is $+1$ the transformation is unperverted, but when Φ_3 is -1 so that one root of the cubic is -1 the transformation will be called perverted. In any transformation which is the product of two reflections there is one line which either remains unchanged both in magnitude and in direction or is merely reversed in direction without change of magnitude. This is the line of intersection of the two planes; and in the former case the transformation is unperverted, while in the latter case it is perverted.

Let the attention be hereafter confined to the unperverted types. The computation of the second of the product gives

$$\Phi_2 = I - 2\zeta\epsilon - 2\xi\eta + 4\xi \cdot \epsilon\zeta\eta.$$

This shows that areas in the plane denoted by $\epsilon \times \eta$, that is, in the plane of the two lines ϵ and η are unchanged. Now the entire result of the transformation may be expressed in terms of the transformation along $\zeta \times \xi$ which is the identical transformation and the transformation in the plane of ϵ and η. Thus it becomes possible to read off the general results in the present case from those obtained previously.* They may be stated as

THEOREM 17. *The product of two planar or two linear reflections which relative to one another have none of the special relations discussed in § 1, is a transformation which leaves the line of intersection of the two planes unchanged whether in magnitude or in direction and which leaves area in the plane of the two lines invariant. The fixed lines of the transformation in this plane divide harmonically the two pairs of lines formed by associating the line of each reflection with the line in which its plane cuts this fixed plane. The transformation is tonic or cyclotonic according as its fixed lines are real or imaginary, that is, according as the two pairs of lines do not or do separate each other. The result for the product of two reflections of which one is planar and the other linear is the same except that the transformation is perverted. And conversely any tonic or cyclotonic which leaves one line invariant in magnitude and direction and which has an invariant plane not passing through this line and subject to the invariancy of areas may be resolved into the product of two reflections both planar or both linear by choosing the invariant line as the intersection of the two planes and by arranging the lines of the reflections and the lines in which their planes cut the fixed plane so as to give the required*

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* If the statements of the following theorem 17 are not entirely clear from the very brief reasoning given here, a reference to my *A generalized conception of area: applications to collineations in the plane*, Annals of Mathematics, ser. 2, vol. 5 (1903), pp. 29–45, will doubtless make them evident. It seems undesirable to increase the dimensions of the present discussion by practically repeating portions of this earlier work.

transformation in this plane.   As this may be done in ∞^1 ways, the above resolution may be accomplished in ∞^1 ways.*   A similar result holds for perverted tonics and cyclotonics.

In the general classification of strains there are seven types if distinction between reals and imaginatives is made.   In the classification of unimodular strains there are these seven types, but the additional relation $\Phi_3 = \pm 1$ limiting the product of the roots of the scalar cubic to ± 1 modifies the result somewhat. If the roots are all alike and if the transformation is real (which has been constantly the point of view of this discussion) the roots are either plus or minus one.   This gives the identical transformation or the central reflection (the perverted identical transformation) if the characteristic equation is $\Phi \mp I = 0$; the simple special shear or its perversion; if the characteristic equation is $(\Phi \mp I)^2 = 0$;   the complex shear or its perversion, if the equation is $(\Phi \mp I)^3 = 0$.   If two of the roots are alike, they must be $+1, +1, -1$ or $+1, -1, -1$ or $a, a, 1/a^2$.   The first two cases give the planar and linear reflections respectively, if the characteristic equation is $(\Phi \pm I)\cdot(\Phi \mp I) = 0$; the simple shear, if the equation is

$$(\Phi + I)\cdot(\Phi - I)^2 = 0 \qquad \text{or} \qquad (\Phi - I)\cdot(\Phi + I)^2 = 0.$$

Finally if the roots are $\pm 1, k, 1/k$ the transformation is a tonic or cyclotonic resoluble into two reflections, and if the roots are not of this type the result is a tonic or cyclotonic not so resoluble.

It should be noted that theorem 16 states that if a transformation is resoluble into two reflections

$$\Phi_{2s} = \mp \Phi_s, \qquad \Phi_3 = \mp 1,$$

according as it is perverted or unperverted.   Now conversely if these relations hold, the cubic becomes a reciprocal equation and the roots cannot be of the form

$$a, a, \pm 1/a^2 \qquad \text{or} \qquad a, b, c \text{ (not } \pm 1, k, 1/k).$$

That is, the roots must bring the transformation under one of the several heads which has already been found to be resoluble into two reflections.   Hence theorem 16 may be completed in

THEOREM 18.   *The necessary and sufficient condition that a strain be resoluble into two reflections is that the invariants of the strain satisfy the relations*

$$\Phi_{2s} = \pm \Phi_s, \qquad \Phi_3 = \pm 1,$$

that is, the scalar cubic shall be reciprocal.

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*It is interesting to note the fact, which might be stated as a theorem, that if a strain is resoluble into two reflections, it is so resoluble in ∞^1, ∞^2, or ∞^4 ways according as its characteristic equation is of the 3rd, the 2nd, or the 1st order.

To return to the geometry relative to the cyclotonic and tonic. The cyclotonic of which the characteristic equation is

$$(\Phi - aI) \cdot (\Phi^2 - 2p \cos q \Phi + p^2 I) = 0$$

may be put in the form *

$$a\alpha' + p \cos q (\beta\beta' + \gamma\gamma') + p \sin q (\gamma\beta' - \beta\gamma').$$

The scalar cubic takes the form

$$(x - a)(x^2 - 2p \cos q \cdot x + p^2) = 0.$$

Now it has been shown that if the transformation is the product of two reflections the scalar cubic may be written as

$$(x \mp 1)[x^2 \mp (\Phi_{2S} - 1)x + 1] = 0.$$

Comparing corresponding coefficients shows that in this case the cyclotonic (confining the attention to the unperverted type) reduces to

$$p = 1, \ a\alpha' + \cos \cos^{-1}[(\Phi_{2S} - 1)/2](\beta\beta' + \gamma\gamma') + \sin \cos^{-1}[(\Phi_{2S} - 1)/2](\gamma\beta' - \beta\gamma').$$

This is a *cyclic* dyadic and represents the transformation which has been called an elliptic rotation through the angle $\cos^{-1}[(\Phi_{2S} - 1)/2]$. If $m = 2\pi/\cos^{-1}[(\Phi_{2S} - 1)/2]$ this dyadic may be regarded as the mth root of the idemfactor.† Hence

THEOREM 19. *The necessary and sufficient condition that a cyclotonic be resoluble into the product of two reflections is that it be merely cyclic. The angle of the elliptic rotation is* $\cos^{-1}[(\Phi_{2S} - 1)/2] = \cos^{-1}(2\zeta \cdot \eta\xi \cdot \epsilon - 1).$

THEOREM 20. *Any root ‡ of the idemfactor may be written in ∞^1 ways as the product of two square roots.*

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* *Vector Analysis*, pp. 355, 361, 366.

† *Vector Analysis*, pp. 348–350.

‡ It has not been shown that even if the cyclic dyadic is a root of the idemfactor, every root must conversely be of this type. To do this proceed as follows: First the determinant of the idemfactor is unity, and the determinant of a pth root of the factor must therefore be a pth root of unity. If the strain is to be real, this pth root must be ± 1 (we may consider p as a prime, if we desire, and then the root would become $+1$). Next the equation $(\sqrt[p]{I})_2 = (\sqrt[p]{I})_3(\sqrt[p]{I})_c^{-1}$ shows that the values of $(\sqrt[p]{I})_{2S}$ and $(\sqrt[p]{I})_S$ are such as to make the scalar cubic reciprocal. Now if the roots of this equation are all real they must be all equal to 1, and there must be a complete set of fixed directions. Hence the root of the idemfactor reduces to $\pm I$, according as p is odd or even. But if two of the roots of the scalar cubic are conjugate imaginaries the pth root of I takes on cyclic form either unperverted or perverted. Thus it is proved that every root of the idemfactor is of this type.

It is not difficult to write down the ellipse in which the rotation takes place. It will be more convenient, as the rest of the work is in space, to write down the cylinder which has this ellipse for director curve and the line of intersection of the two planes for generator. The three directions ϵ, $\zeta \times \xi$, $(\epsilon \times \eta) \times \zeta$, of which the last represents the line of intersection of the ζ-plane with the plane of ϵ and η, are three conjugate directions in the quadric surface.* The three directions reciprocal to these are ζ, $\epsilon \times (\zeta \times \xi)$, $\epsilon \times \eta$ and the quadric may therefore be written in the form †

$$a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi) + c\epsilon \times \eta\epsilon \times \eta.$$

The condition that the quadric be a cylinder parallel to $\zeta \times \zeta$ necessitates the vanishing of c, and the fact that η and $\xi \times (\epsilon \times \eta)$ are conjugate directions gives

$$\eta \cdot [a\zeta\zeta + b\epsilon \times (\zeta \times \xi)\epsilon \times \cdot (\zeta \times \xi)] \cdot \xi \times (\epsilon \times \eta) = 0.$$

If the ratio of a to b is determined from this condition, the final form of the cylinder turns out to be

$$\rho \cdot [(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \xi\zeta\zeta + \eta \cdot \zeta\epsilon \times (\zeta \times \xi)\epsilon \times (\zeta \times \xi)] \cdot \rho = \text{const.}$$

The above deduction of the equation of the conic in which the points of planes parallel to the plane of the two lines ϵ and η move is in no wise restricted to the case of elliptic rotation. If the transformation were a tonic which leaves area invariant the motion of the points of the plane would be in hyperbolas, and the above equation would be that of the hyperbolic cylinders of which the director curves are these hyperbolas and of which the elements are parallel to $\zeta \times \xi$. In fact the form of the equation shows that the type is elliptic or hyperbolic according as

$$(1 - \epsilon \cdot \xi\zeta \cdot \eta)\epsilon \cdot \zeta \qquad \text{and} \qquad \zeta \cdot \eta$$

have the same or opposite signs — which is equivalent to the criterion established to distinguish between the cyclotonic and tonic. This suggests the question of whether it may not be possible to explain the transformation in case it is a tonic as a sort of hyperbolic rotation, and thus introduce a greater symmetry into the treatment of the tonic and cyclotonic.

It should be remembered that in elliptic rotation the dyadic

$$\Phi = \alpha\alpha' + \cos q(\beta\beta' + \gamma\gamma') + \sin q(\gamma\beta' - \beta\gamma')$$

used as a prefactor advances a radius vector in an ellipse of which β and γ are

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* This follows from the harmonic property stated in theorem 17 and from the harmonic properties of conjugate directions.

† *Vector Analysis*, p. 378.

conjugate diameters through a sector, of which the area is to the area of the whole ellipse as q is to 2π.* In order to extend this statement to the case of the hyperbola it is necessary to replace the area of the ellipse by the area of the rectangle constructed on the semi-axes of the ellipse. Then the radius vector advances through a sector, the area of which is to the area of the rectangle as q is to 2. From analogy we state

THEOREM 21. *A dyadic of the form*

$$\Phi = \alpha\alpha' + \cosh q\,(\beta\beta' + \gamma\gamma') + \sinh q\,(\gamma\beta' + \beta\gamma')$$

used as a prefactor advances a radius vector in a hyperbola of which the vectors β and γ are conjugate semi-diameters through a sector, of which the area is to the area of the rectangle constructed on the semi-axes (or the parallelogram on any conjugate semi-diameters) as q is to 2.

As the hyperbola of which β and γ are conjugate radius vectors may be projected into a rectangular hyperbola in such a manner that the said conjugate direction become the axes of the rectangular hyperbola and all areas remain the same, and as the statement of the problem admits a transformation of similitude with the origin as center, it will be sufficient to prove the theorem in the case of the hyperbola $x^2 - y^2 = 1$ and the dyadic

$$\mathbf{ii} + \cosh q\,(\mathbf{jj} + \mathbf{kk}) + \sinh q\,(\mathbf{kj} + \mathbf{jk}).$$

Any radius vector in the hyperbola may be written as

$$\rho = \mathbf{i}\cosh p + \mathbf{j}\sinh p$$

and then

$$\rho = \Phi\cdot\rho = \mathbf{i}\cosh(p+q) + \mathbf{j}\sinh(p+q).$$

The area of the sector of the hyperbola from the horizontal to any inclination is

$$\tfrac{1}{2}\int_0^\theta \rho\cdot\rho\,d\theta = \tfrac{1}{2}\int_0^p (\cosh^2 p + \sinh^2 p)\,d\tan^{-1}\tanh p = \tfrac{1}{2}p.$$

Hence the theorem is proved. It might be noted that this tonic has the analytic form of a $2\pi/(q\sqrt{-1})$th root of the idemfactor.

The criterion for a versor† is $\Phi\cdot\Phi_C = I$, $\Phi_3 = 1$. To apply this test to ascertain under what conditions the product of two reflections is a versor would be less simple than to proceed from the point of view of the elliptic rotation. If the ellipse becomes a circle the rotation becomes versorial, and conversely.

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* *Vector Analysis*, p. 349.

† *Vector Analysis*, p. 335.

Hence for the versor ϵ, η, ζ, ξ lie in a plane perpendicular to the axis of version. Moreover as the conjugate diameters of a circle are perpendicular to each other, the lines of each reflection must be normal to its plane of the reflection. There are similar results for the perversor. Hence

THEOREM 22. *The necessary and sufficient condition that the product of two reflections be a versor is that the reflections be orthogonal and of the same type ; if one is planar and the other linear, the result is a perversor.*

The actual expression for the elliptic or hyperbolic rotation has not yet been given in canonical form in terms of the elements which enter into the reflections of which the transformation is compounded. There is no difficulty in doing this. In fact the vectors which correspond to the α, β, γ of the canonical form may be chosen as $\zeta \times \xi$, ϵ, $(\epsilon \times \eta) \times \zeta$. If m denote the quantity $\xi \cdot \epsilon \zeta \cdot \eta$ the reciprocals are

$$\alpha' = \frac{\epsilon \times \eta}{m-1}, \qquad \beta' = \zeta, \qquad \gamma' = \frac{\epsilon \times (\zeta \times \xi)}{m-1}.$$

From these the canonical form may be written down whether for the elliptic or hyperbolic type by following the models on pages 291–3. The fixed lines of the transformation which are the asymptotes of the ellipse or hyperbola are along the directions

$$\eta + [\sqrt{(m-1)m} \pm m] \epsilon.$$

We pass on to the case of the resolution of the collineation or strain with a fixed point and a unit modulus into three reflections. Let Ω represent the strain. If the modulus is $\Omega_3 = +1$, all three of the reflections are linear or two of them are planar and the third is linear ; whereas if the modulus is $\Omega_3 = -1$, two are linear and one is planar or all three are planar. Suppose the case to be that of the first of these four. It is required to find a reflection $2\epsilon\zeta - I$ such that

$$[\Omega \cdot (2\epsilon\zeta - I)]_{2S} = [\Omega \cdot (2\epsilon\zeta - I)]_S.$$

By use of the relation $(\Omega \cdot \Phi)_2 = \Omega_2 \cdot \Phi_2$ this reduces to

$$2\epsilon \cdot \Omega_2 \cdot \zeta - \Omega_{2S} = 2\epsilon \cdot \Omega_C \cdot \zeta - \Omega_S.$$

But we have always $\Omega_3 I = \Omega_C \cdot \Omega_2$, and hence $\Omega_2 = \Omega_C^{-1}$. That is to say,

$$2\zeta \cdot (\Omega - \Omega^{-1}) \cdot \epsilon = \Omega_S - \Omega_S^{-1}, \qquad \Omega_S^{-1} = (\Omega^{-1})_S \neq (\Omega_S)^{-1},$$

is the equation which must be satisfied by ϵ and ζ.

The other three possible cases may be reduced to these. For in the second case Ω_3 is still equal to $+1$, but the first reflection applied may be planar.

Then the product $\Omega \cdot (I - 2\epsilon\zeta)$ would have a negative modulus and the condition that it be resoluble into two reflections would be

$$[\Omega \cdot (I - 2\epsilon\zeta)]_{2S} = - [\Omega \cdot (I - 2\epsilon\zeta)]_S,$$

which is precisely the same as before. If on the other hand $\Omega_3 = -1$, we may effect the resolution of $-\Omega$ as above. But the reciprocal of $-\Omega$ is $-\Omega^{-1}$. Thus the condition is unchanged. The form of the condition may be changed by noting that

$$\zeta \cdot \epsilon = \zeta \cdot I \cdot \epsilon = 1 \qquad \text{and} \qquad \Omega_S - \Omega_S^{-1} = \zeta \cdot (\Omega_S - \Omega_S^{-1}) I \cdot \epsilon$$

and hence

$$\zeta \cdot [(\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_S - \Omega_S^{-1}) I] \cdot \epsilon = 0.$$

We may therefore state

THEOREM 23.   *In order that ϵ and ζ may serve as line and plane, respectively, for the first of three reflections into which a strain Ω may be resolved, it is necessary and sufficient that they satisfy the equation*

$$\zeta \cdot \Psi \cdot \epsilon = 0, \qquad \Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_S - \Omega_S^{-1}) I.$$

First consider the cases in which Ω is itself resoluble into two reflections. Here

$$\Omega_{2S} = \Omega_3 \Omega_{CS}^{-1} = \Omega_3 \Omega_S^{-1} = \pm \Omega_S \qquad \text{and hence} \qquad \Omega_S - \Omega_S^{-1} = 0,$$

and the condition reduces to

$$\zeta \cdot (\Omega - \Omega^{-1}) \cdot \epsilon = 0.$$

The dyadic $\Omega - \Omega^{-1}$ is then always incomplete; for

$$(\Omega - \Omega^{-1})_3 = \Omega_3 - \Omega_2 : \Omega^{-1} + \Omega : \Omega_2^{-1} - \Omega_3^{-1} = \Omega : \Omega_2^{-1} - \Omega_2 : \Omega^{-1}.$$

But

$$\Omega : \Omega_2^{-1} = (\Omega \cdot \Omega_{2C}^{-1})_S = (\Omega^2)_S \qquad \text{and} \qquad \Omega_2 : \Omega^{-1} = (\Omega_2^2)_S.$$

Hence the question resolves itself into the question whether the square of a transformation resoluble into two reflections is itself so resoluble. The answer is affirmative, as may be seen by investigating the squares of the various types that have been obtained: but though this method leads to a number of interesting relations, it is too long to take up here. The general expression for any dyadic may be written * as

$$\Omega = a\alpha\alpha' + g\alpha\beta' + f\alpha\gamma' + b\beta\beta' + e\beta\gamma' + c\gamma\gamma'$$

in a triply infinite system of ways. The coefficients of the diagonal terms are

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* This general form may be recognized at once by checking it against the various types we have noted as possible in the classification of strains. It is, however, merely the special case in three dimensions of a general form to which a collineation may be reduced. This form is not so

the roots of the scalar cubic. The square of this dyadic is, as far as concerns the terms in the diagonal,

$$\Omega^2 = a^2\alpha\alpha' + b^2\beta\beta' + c^2\gamma\gamma' + 3 \text{ terms.}$$

Now if one of the quantities a, b, c is numerically unity and the other two are reciprocals, the same is true of a^2, b^2, c^2. Hence

THEOREM 24. *If Ω is resoluble into the product of two reflections the dyadic $\Omega - \Omega^{-1}$ is incomplete, and the condition on ϵ and ζ becomes*

$$\zeta \cdot \Phi \cdot \epsilon = 0, \qquad \Phi = \Omega - \Omega^{-1}, \qquad \Phi_S = 0,$$

and conversely.

It is possible to write $\Omega - \Omega^{-1}$ as $\Omega^{-1} \cdot (\Omega^2 - I)$, and all the degrees of incompleteness must occur in the factor $\Omega^2 - I$. If this vanishes, Ω is itself a reflection; if it is linear, all three roots of the cubic are numerically unity and the transformation is one of the simple shears, special or non-special; if it is merely planar, the transformation is a complex shear or an elliptic or hyperbolic rotation. In the first of these three cases ϵ may be chosen at pleasure and so may ζ. The condition is always satisfied. In the second case, if ϵ is taken perpendicular to the consequent of Φ, ζ may be chosen at pleasure, whereas if ϵ is not so chosen, ζ must be selected from the vectors perpendicular to the antecedent of Φ and not perpendicular to ϵ. It would be impossible to make this choice if ϵ had been chosen parallel to the antecedent of Φ, that is, if $\Phi \cdot \epsilon = x\epsilon$ and $x \neq 0$. But in this case, as Φ is linear, $\Phi \cdot \epsilon = \epsilon \Phi_S = 0$. Hence any choice of ϵ is admissible. In the last case any choice for ζ (not perpendicular to ϵ) is permissible if ϵ be taken perpendicular to the plane of the consequents of Φ. But if ϵ be not so taken, then ζ is restricted to plane perpendicular to $(\Omega - \Omega^{-1}) \cdot \epsilon$. There may be two independent values for ϵ, for which $(\Omega - \Omega^{-1}) \cdot \epsilon$ becomes parallel to ϵ without vanishing, and the indicated resolution then becomes impossible. A brief examination of this case shows that the

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widely used as it would be if authors were less willing to assume the "general case" of a complete system of fixed points when discussing collineations. (It may be noted in passing that the group of translations does not contain a single transformation which can be said to belong to the "general case.") The general form to which any collineation may be reduced is

$$\rho x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$\rho x_2' = \qquad\quad a_{22}x_2 + \cdots + a_{2n}x_n \quad \rho x_{n-1}' = a_{n-1\ n-1}x_{n-1} + a_{n-1\ n}x_n$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \rho x_n' = \qquad\qquad a_{nn}x_n$$

where only the terms on one side of the diagonal occur. The proof of the reduction is easy. The collineation has one fixed linear space S_{n-1}. Take this as the space $x_n = 0$ of reference. In this space there must be one fixed subspace S_{n-2}. Choose the second space $x_{n-1} = 0$ of reference to pass through this space S_{n-2}. In the space S_{n-2} there is likewise a subspace S_{n-3} which is fixed. Pass the third space $x_{n-2} = 0$ of reference through this S_{n-3}. And so on. It is not hard to show that with this choice of spaces of reference the collineation reduces to the type here given.

only directions ϵ for which $(\Omega - \Omega^{-1}) \cdot \epsilon$ is parallel to ϵ without vanishing are the fixed directions of the rotations. These results and some others which it is not difficult to verify may be stated in the following theorems:

THEOREM 25. *An elliptic or hyperbolic rotation may be resolved into three reflections by taking the line of the first reflection at random (with the exception of the two fixed lines in the plane of the rotation) and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. There is one exceptional choice of the line and one of the plane such that the choice of the other remains arbitrary.*

THEOREM 26. *A complex shear may be resolved into three reflections by taking the line of the first reflection at random and taking the plane through a specified line, etc., as in theorem 25.*

THEOREM 27. *A simple shear of the types considered above, whether special or non-special, may be resolved into the product of three reflections by taking the line of the first reflection at random and taking the plane through a specified line; or by assuming the plane of the first reflection at random and choosing the line in a specified plane. If, exceptionally, the line be chosen in a certain plane or the plane through a certain line, the subsequent choice of the other remains arbitrary.*

In the discussion of the case where Ω is not resolvable into two reflections, the behavior of the dyadic

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_S - \Omega^{-1}_S)I$$

is of fundamental importance. If Ω be expressed in the reduced form

$$\Omega = a\alpha\alpha' + b\beta\beta' \pm \frac{1}{ab}\gamma\gamma' + 3 \text{ terms}$$

the dyadic Ψ takes the form

$$-\Psi = \left[\tfrac{1}{2}\left(a + b \pm \frac{1}{ab} - \frac{1}{a} - \frac{1}{b} \mp ab\right) - \left(a - \frac{1}{a}\right)\right]\alpha\alpha'$$

$$+ [\quad]\beta\beta' + [\quad]\gamma\gamma' + 3 \text{ terms,}$$

and will be complete unless one of the coefficients of the three diagonal terms vanishes. Suppose the first one to vanish. Then

$$-\left(a - \frac{1}{a}\right) + \left(b - \frac{1}{b}\right) \pm \left(\frac{1}{ab} - ab\right) = 0$$

or

$$-a^2b + b + ab^2 - a \pm 1 \mp a^2b^2 = 0.$$

This expression is always factorable into three factors chosen from among the six $(b \pm 1)(a \pm 1)(ab \pm 1) = 0$. These are just the conditions that one of the roots be numerically unity, that is, that the strain Ω be resolvable into two reflections. Hence

THEOREM 28.   *The necessary and sufficient condition that* Ω *be resoluble into two reflections is that*

$$\Psi = (\Omega - \Omega^{-1}) - \tfrac{1}{2}(\Omega_s - \Omega_s^{-1})I$$

be incomplete.

In case Ω is not resoluble into two reflections, it is (by virtue of this theorem) impossible to find any direction for the line ϵ which shall allow the plane ζ to be arbitrary.   The question arises whether there may not be directions for ϵ which shall make it impossible to find any ζ whatsoever.   The case of impossibility can only arise when $\Psi \cdot \epsilon$ is parallel to ϵ.   That is, in case ϵ is one of the fixed directions of Ψ.   But as every direction is fixed in the transformation kI, it is sufficient to state that ϵ is a fixed direction for $\Omega - \Omega^{-1}$, and a reference to the expression of this dyadic in reduced form will show that this is equivalent to saying that ϵ is a fixed direction of Ω.   Now there are only two types of strain which we have to consider and which are not resoluble into two reflections. They are where the roots are a, a, $\pm 1/a^2$, which is a simple shear or special tonic, and where the roots are a, b, c and none of them numerically unity — the tonic or cyclotonic.   In the case of the tonic or cyclotonic there are just three directions of impossibility (and for the cyclotonic two of these are imaginary), for the simple shear there are two such directions of which one is the fixed direction of the shear and the other the direction parallel to which the shearing takes place; for the special tonic there is a whole plane of such directions, the plane of the equal roots a, a.

THEOREM 29.   *Any unimodular strain not resoluble into two reflections is resoluble into three reflections of which the first has an arbitrary line (provided it be not collinear with any of the fixed directions of the strain) and a plane which must pass through a definite line but is otherwise unlimited except that it should not pass through the line of the reflection.*

It may be observed in general as a result of the foregoing theorems 25–29 that to resolve a strain into three reflections we have

THEOREM 30.   *In all cases the line of the first reflection may be chosen at random (except along such fixed directions of the strain as correspond to stretching) and the plane may be any plane which passes through a specified line but does not contain the line of the reflection.*

The proportions which this article has attained and this natural close make it unwise to continue here and now with the more detailed relations of oblique reflections to the unimodular strains.

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